Math 680: Geometry of Curves and Surfaces

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Fall 2020

About This Course

This course is an introduction to Differential Geometry, and serves as the basis for the intro graduate sequence in Geometry and Topology. It was taken in the Fall of 2019 at UNC Chapel Hill, and taught by Professor Yaiza Canzani. We used Barrett O'Neill's text Elementary Differential Geometry. These notes were copied from the ones in my notebook and any mistakes are mine and not the lecturers.

Contents

1 Curves

Definition. A tangent vector at p, v_p in $\mathbb{R}^n$ is a pair of points (p, v) so that $v_p = v + p$. We call v the direction and p the base point

Definition. Let $p \in \mathbb{R}^n$ then $T_p\mathbb{R}^n$ the the tangent space to $\mathbb{R}^n$ at p and is defined as follows

$$T_p\mathbb{R}^n = \{v_p : v_p \text{ is a tangent vector at } p\}$$

Note that $T_p\mathbb{R}^n$ is a vector space in via the following operations:

$$v_p + w_p = (v + w)_p, \qquad av_p = (av)_p$$

For tangent vectors v, w based at p and a real number a. Moreover there is a vector space isomorphism

$$T_p\mathbb{R}^n \cong \mathbb{R}^n$$

$$v_p \mapsto v$$

Definition. A vector field V on $\mathbb{R}^n$ is a map that to each point $p \in \mathbb{R}^n$ associates a tangent vector

$$V(p) \in T_p\mathbb{R}^n$$

Observation. Vector fields form an algebra

- $(V + W)(p) = V(p) + W(p)$
- $(fV)(p) = f(p)V(p)$

Where $p \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$

Definition. Natural frame field or basis, is defined as follows:

$$U_i(p) = (0, ..., 1, ..., 0) \in T_p\mathbb{R}^n$$

From this we can define a vector field coordinate wise:

$$V(p) = (V_1, ..., V_n(p)) = \sum_{i=1}^{n} v_i(p)U_i(p)$$

What is the tangent space to a curve? Well just a line, a la Calculus. What about a sphere? Here it's a plane. Now we have a problem. We don't want tangent vectors to change as we move dimensions, but our current definition depends on the dimension of our space.

Instead of thinking of tangent vectors as points, we will think of them as operators: maps that act on the space of functions. Via this we have a new (read: correct) definition of tangent vector

Definition. A tangent vector is

$$v_p(f) = \frac{d}{dt} f(p + tv)\bigg|_{t=0}$$

Which is the derivative of f in the direction of v_p for $f : \mathbb{R}^n \to \mathbb{R}$

Observation.

$$V_p(f) = \frac{d}{dt} f(p_1 + tv_1, ..., p_n + tv_n)\Big|_{t=0} = \sum_{i=1}^{n} v_i \left.\frac{\partial}{\partial x_i}\right|_p (f) = \langle v, \nabla f(p) \rangle$$

$$v_p = \sum_{i=1}^{n} v_i \left.\frac{\partial}{\partial x_i}\right|_p$$

This is an intrinsic way to define a tangent vector at a point p, invariant of choice of coordinates. The partial derivatives are a notational way to define the basis of $T_p\mathbb{R}^n$

Example. $f(x,y,z) = x^2yz$, we want to find v_p for $v = (1,0,-3), p = (1,1,0)$ then we have $v_p = \langle v, \nabla f(p) \rangle$. So

$$\nabla f(x,y,z) = \langle 2xyz, x^2z, x^2y \rangle$$
$$\nabla f(1,1,0) = \langle 0,0,1 \rangle$$

Hence

$$v_p(f) = \langle (1,0,-3), (0,0,1) \rangle = -3$$

The tangent vector measures how the function goes up/down, in our case the graph is decreasing -3 at each step of p.

Proposition. Let $a, b \in \mathbb{R}, f, g : \mathbb{R}^n \to \mathbb{R}, v_p, u_p \in T_p\mathbb{R}^n$

- $(av_p + bu_p)(f) = av_p(f) + bu_p(f)$
- $v_p(af + bg) = av_p(f) + bv_p(f)$
- $v_p(fg) = f(p)v_p(g) + g(p)v_p(f)$

Vector fields have the same properties as above

We now begin our study of curves

Definition. A curve in $\mathbb{R}^n$ is a map $\alpha : I \to \mathbb{R}^n$ with interval I, that will be smooth.

Definition. If α is smooth, the velocity of α at time t is

$$\alpha'(t) = (\alpha_1'(t), ..., \alpha_n'(t))_{\alpha(t)} \in T_{\alpha(t)}\mathbb{R}^n$$

Said a different way:

$$\alpha'(t) = \sum_{i=1}^{n} \alpha_i'(t) U_i(t)$$

Definition. Let $\alpha : I \to \mathbb{R}^n$ be a differentiable curve, and $h : J \to I$ be a differentiable function. We define the reparameterization of α to be the curve

$$\beta(t) = (\alpha \circ h)(t)$$

Definition. $\{e_1, ..., e_n\} \in T_p\mathbb{R}^n$ is a frame at the point p if the vectors are linearly independent, orthogonal to each other and unit length.

Proposition. Let $v_p \in T_p\mathbb{R}^n$ and $\{e_1, ..., e_n\}$ be a frame at p. Then

$$v_p = \sum_{i=1}^{n} \langle v_p, e_i \rangle e_i$$

We can take cross products of tangent vectors as usual. We also have the following:

Observation.

- $||v \times w||^2 = ||v||^2||w||^2$
- $||v \times w|| = ||v||||w|| \sin\theta$
- $\langle v, w \rangle = ||v||||w|| \cos\theta$

Definition. Let $\alpha : I \to \mathbb{R}^n$ be a smooth curve, then the speed at time t is $||\alpha'(t)||$

Definition. If $a, b \in I, a < b$ then Length$([a, b]) = \int_a^b ||\alpha'(t)|| \, dt$

Definition. $\alpha : I \to \mathbb{R}^n$ is said to be parameterized by arc length or is unit speed if $||\alpha'(t)|| = 1, \forall\, t$

Theorem. Let $\alpha : I \to \mathbb{R}^n$ be a differentiable curve such that

2 Differential Forms

3 Shape Operator

The shape operator will act somewhat like a Frenet frame for an abstract surface. Let M be a surface, and Z be a vector field. Fix $p \in M$ and $v \in T_pM$.

Definition. The covariant derivative of Z in the direction of v is

$$\nabla_v Z = (Z \circ \alpha)'(0)$$

Where $\alpha(0) = p, \alpha'(0) = v$.

This tells us how the vector field Z is changing in the direction of α. Here are a few formulations. Let $Z = \sum z_i U_i$, then

$$\begin{aligned}
\nabla_v Z &= (Z \circ \alpha)'(0) \\
&= \left(\sum_{i=1}^{3} Z_i(\alpha(t)) U_i(\alpha(t)) \right)' \Bigg|_{t=0} \\
&= \sum_{i=1}^{3} Z_i(\alpha(t))' U_i(\alpha(t)) \Bigg|_{t=0} \\
&= \sum_{i=1}^{3} v[z_i](p) U_i(p)
\end{aligned}$$

Definition. The shape operator if a map that to each $p \in M$ associates a linear map S_p such that

$$S_p : T_pM \to T_pM$$

$$S_p(v) = -\nabla_v U$$

For a unit normal vector field U to M.

Example. For a plane the shape operator is 0. Conceptually the covariant derivative measures how much U is changing along v, which is 0 for a plane.

Example. For a cylinder it depends on the direction. For vectors on the "north to south" direction $S_p(v_2) = 0$. For vectors around the circle the shape operator is $S_p(v_1) = -v_1$

Example. For a sphere the shape operator is -Id